Modeling The Business Cycle
Part I - Cyclical Revenue

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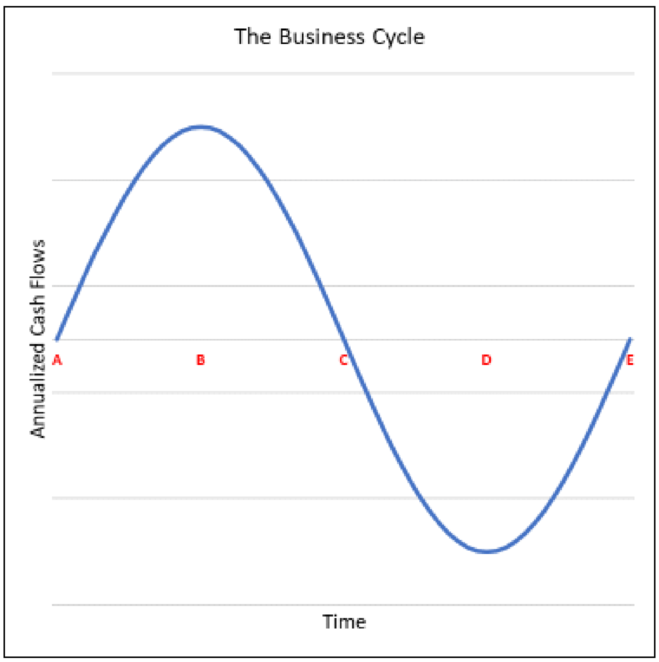
The business cycle, also known as the economic cycle, is the downward and upward movement of gross domestic product (GDP) around its long-term growth trend. The length of a business cycle is the period of time containing a single boom and contraction in sequence. This cycle is generally separated into four distinct segments, expansion, peak, contraction, and trough. The figure below is a graphical example of the business cycle...

Figure 1: The Business Cycle



Business Cycle Segments:

Point A - Beginning of the business cycle

Point A to B - Expansion phase of the business cycle.

Point B - Peak of the business cycle.

Point B to D - Contraction phase of the business cycle.

Point D - Trough of the business cycle.

Point D to E - Expansion phase of the business cycle.

Point E - End of the business cycle.

A cyclical industry refers to a type of industry whose revenue generation capabilities are tied to the business cycle. The following industries are commonly classified as cyclical: Auto components, construction, semiconductor, steel, airline, hotels/restaurants/leisure, and textile/apparel/luxury goods.

The problem from a valuation standpoint is that if we use cashflows around the peak of the cycle then that company will be overvalued and if we use cashflows around the trough of the cycle then that company will be undervalued. In this white paper we will build a model that properly accounts for the business cycle. To that end we will work throught the following hypothetical problem...

Our Hypothetical Problem

We are tasked with forecasting revenue for ABC Company. ABC Company's revenue is cyclical and the company is currently at the top of the business cycle. It is projected that ABC Company's annualized revenue would decrease by 50% from the business cycle peak-to-trough (assumes a secular revenue growth rate of zero). The table below presents ABC Company's go-forward model assumptions...

Table 1: Model Assumptions

Description	Balance	Notes
Annualized revenue at time zero	$10,000,000	Current revenue annualized
Annualized revenue growth rate (%)	5.00	Discrete-time secular growth rate (RGR)
Annualized revenue volatility (%)	25.00	Secular growth rate standard deviation
Cost of capital (%)	12.00	Discrete-time discount rate
Peak-to-trough change in revenue (%)	50.00	Excludes secular growth over time interval
Business cycle length in months	60	

We are tasked with answering the following questions:

Question: What is expected cumulative revenue in year three?

Modeling The Business Cycle

We will define the variable θ_t to be the number of radians at time t, the variable ω to be the length of one business cycle in years, and the variable ϕ to be the current position in the business cycle in years. The equation for the number of radians at time t is...

$$\theta_t = \beta\,(t + \phi) \text{ ...where... } \beta = \frac{2\,\pi}{\omega} \tag{1}$$

We will define the variable Δ to be the business cycle amplitude, which is defined as one-half of the distance from peak to trough. Using Equation (1) above and the data from Table 1 above selected model parameters for our hypothetical problem are...

$$\omega = \frac{60}{12} = 5 \text{ ...and... } \phi = \frac{15}{12} = 1.25 \text{ ...and... } \Delta = \frac{0.50}{2} = 0.25 \text{ ...and... } \beta = \frac{2\,\pi}{5} = 1.2566 \tag{2}$$

We will define the variable Γ_t to be the company-specific business cycle factor at time t. Using Equations (1) and (2) above the equation for the business cycle factor is...

$$\Gamma_t = \Delta\,\sin(\theta_t) \tag{3}$$

Using Equations (1), (2) and (3) above the graphs of ABC Company's theta value and business cycle factor are...

Figure 2: ABC Company Theta Value

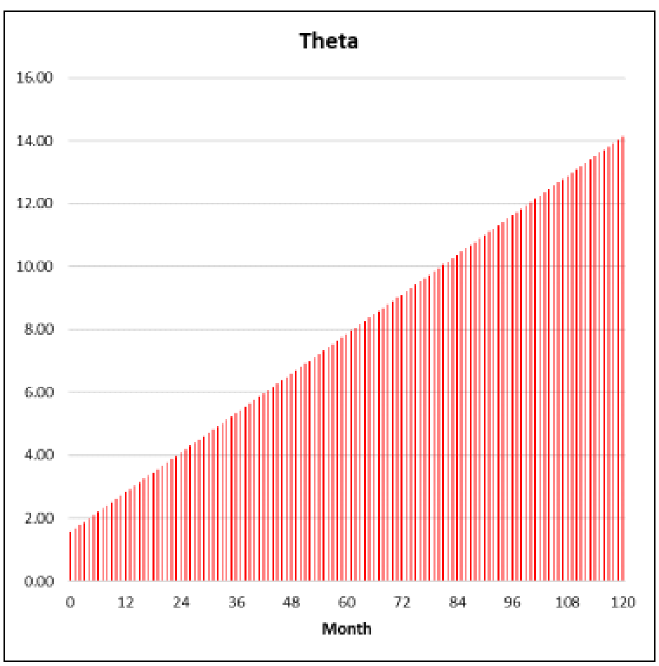
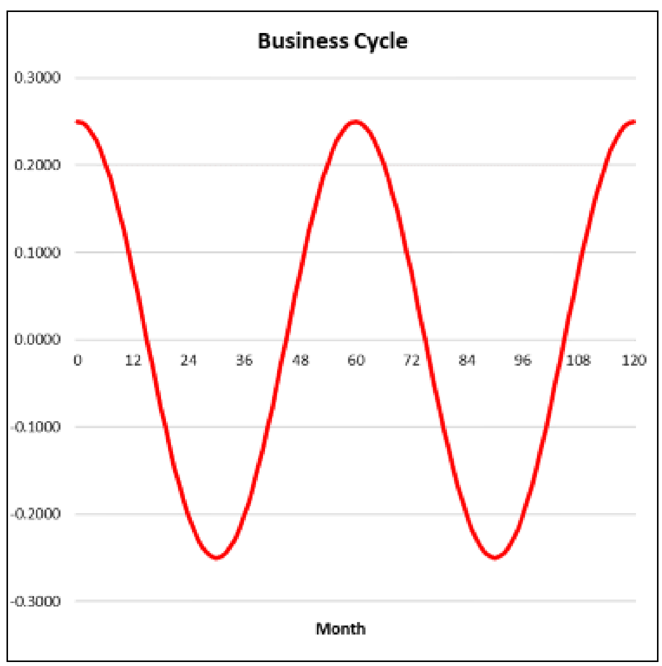


Figure 3: ABC Company Business Cycle Factor



Annualized Revenue

We will define the variable Y_t to be random non-cyclical annualized revenue at time t, the variable λ to be the continuous-time secular revenue growth rate, the variable σ to be revenue growth rate volatility, and the variable Z_t to be a normally-distributed random variable with mean zero and variance one. The stochastic differential equation that defines the change in non-cyclical annualized revenue over time is... [2]

$$\delta Y_t = \lambda\, Y_t\, \delta t + \sigma\, Y_t \sqrt{t}\, Z_t \;\; \text{...where...} \;\; \lambda = \ln\left(1 + RGR\right) \;\; \text{...and...} \;\; Z_t \sim N\left[0,1\right] \tag{4}$$

The solution to Equation (4) above is the equation for random non-cyclical annualized revenue at time t. The equation for random non-cyclical annualized revenue is... [2]

$$Y_t = Y_0\,\mathrm{Exp}\left\{\left(\lambda - \frac{1}{2}\,\sigma^2\right)t + \sigma\,\sqrt{t}\,Z_t\right\} \tag{5}$$

Given that returns are normally-distributed random non-cyclical annualized revenue in the equation above is a lognormally-distributed random variable. Using Equation (5) above the equation for expected non-cyclical annualized revenue at time t is... [1]

$$\mathbb{E}\left[Y_t\right] = Y_0\,\mathrm{Exp}\left\{\lambda\, t\right\} \tag{6}$$

We will define the variable R_t to be random cyclical annualized revenue at time t. Using Equations (1), (3) and (5) above the equation for random cyclical annualized revenue at time t as a function of non-cyclical annualized revenue at time zero is... [2]

$$R_t = Y_t\left(1 + \Delta\,\Gamma_t\right) = Y_0\left(1 + \Delta\,\sin(\beta\,(t+\phi))\right)\mathrm{Exp}\left\{\left(\lambda - \frac{1}{2}\,\sigma^2\right)t + \sigma\,\sqrt{t}\,Z_t\right\} \tag{7}$$

Using Equation (7) above the equation for expected cyclical annualized revenue at time t is... [1]

$$\mathbb{E}\left[R_t\right] = \left(1 + \Delta\,\Gamma_t\right)\mathbb{E}\left[Y_t\right] = Y_0\left(1 + \Delta\,\sin(\beta\,(t+\phi))\right)\,\mathrm{Exp}\left\{\lambda\, t\right\} \tag{8}$$

Whereas actual cyclical annualized revenue at time zero (R_0) can be observed non-cyclical annualized revenue at time zero (Y_0) cannot. Using Equation (8) above we can make the following statement...

$$R_0 = Y_0\left(1 + \Delta\,\sin(\beta\,(0+\phi))\right)\,\mathrm{Exp}\left\{\lambda \times 0\right\} = Y_0\left(1 + \Delta\,\sin(\beta\,\phi)\right) \;\; \text{...when...} \;\; t = 0 \tag{9}$$

Using Equation (9) above and solving for the unobservable variable Y_0 we get the following equation...

$$Y_0 = R_0\left(1 + \Delta\,\sin(\beta\,\phi)\right)^{-1} \tag{10}$$

Using Equation (10) above we can rewrite Equation (7) above as...

$$R_t = R_0\left(1 + \Delta\,\sin(\beta\,\phi)\right)^{-1}\left(1 + \Delta\,\sin(\beta\,(t+\phi))\right)\mathrm{Exp}\left\{\left(\lambda - \frac{1}{2}\,\sigma^2\right)t + \sigma\,\sqrt{t}\,Z_t\right\} \tag{11}$$

Using Equation (10) above we can rewrite Equation (8) above as...

$$\mathbb{E}\left[R_t\right] = R_0\left(1 + \Delta\,\sin(\beta\,\phi)\right)^{-1}\left(1 + \Delta\,\sin(\beta\,(t+\phi))\right)\,\mathrm{Exp}\left\{\lambda\, t\right\} \tag{12}$$

Using the equations above and the data in Table 1 above the graphs of expected and random annualized revenue over the time interval year zero to year ten are...

Figure 4: ABC Company Expected Annualized Revenue (Average of All Paths)

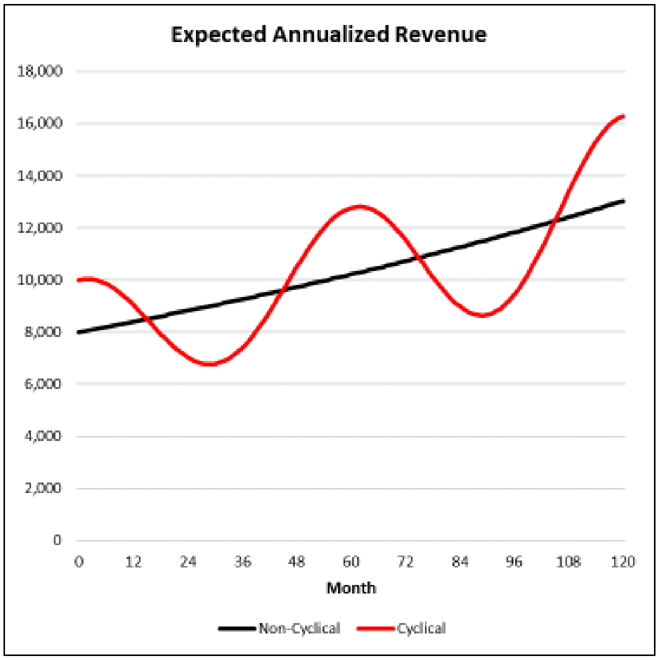
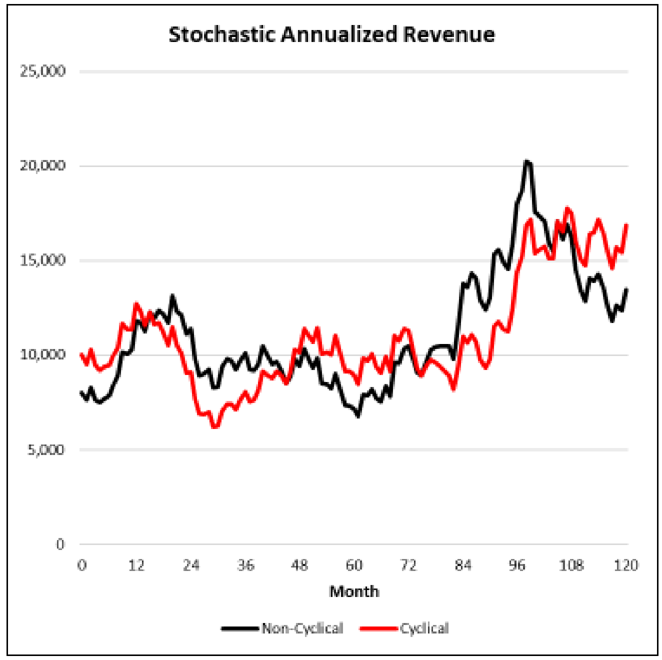


Figure 5: ABC Company Stochastic Annualized Revenue (One Path)



Cumulative Revenue

We will define the variable $C_{a,b}$ to be expected cumulative revenue realized over the time interval $[a, b]$. Using Equations (12) above the equation for expected cumulative cyclical revenue is...

$$C_{a,b} = \int_a^b \mathbb{E}\Big[R_t\Big]\,\delta t = \int_a^b R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(1+\Delta\,\sin(\beta\,(t+\phi))\Big)\,\mathrm{Exp}\Big\{\lambda\,t\Big\}\,\delta t \tag{13}$$

Note that we can rewrite Equation (13) above as...

$$C_{a,b} = R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Bigg(\int_a^b \mathrm{Exp}\Big\{\lambda\,t\Big\}\,\delta t + \Delta\int_a^b \mathrm{Exp}\Big\{\lambda\,t\Big\}\,\sin(\beta\,(t+\phi))\,\delta t\Bigg) \tag{14}$$

Using Appendix Equations (20) and (21) below we can rewrite Equation (14) above as...

$$C_{a,b} = R_0\Big(1+\Delta\,\sin(\beta\,\phi)\Big)^{-1}\Big(I(a,b)_1 + \Delta\,I(a,b)_2\Big) \tag{15}$$

The Answer To Our Hypothetical Problem

Using Equation (2) above and Appendix Equation (20) below the solution to the first integral in Equation (15) above is...

$$I(2,3)_1 = \mathrm{Exp}\Big\{0.0488\times 3\Big\}\,0.0488^{-1} - \mathrm{Exp}\Big\{0.0488\times 2\Big\}\,0.0488^{-1} = 1.1298 \tag{16}$$

Using Equation (2) above and Appendix Equation (21) below the solution to the second integral in Equation (15) above is...

$$I(2,3)_2 = \Bigg[\text{Exp}\Bigg\{0.0488 \times 3\Bigg\}\Bigg(0.0488 \times \sin(1.2566 \times (3 + 1.25)) - 1.2566 \times \cos((1.2566 \times (3 + 1.25))\Bigg) - \text{Exp}\Bigg\{0.0488 \times 2\Bigg\}\Bigg(0.0488 \times \sin(1.2566 \times (2 + 1.25)) - 1.2566 \times \cos((1.2566 \times (2 + 1.25))\Bigg)\Bigg] \times \Bigg(0.0488^2 + 1.2566^2\Bigg)^{-1} = -1.0569 \tag{17}$$

Question: What is expected cumulative revenue in year three?

Using Equations (2), (15), (16) and (17) above and the data in Table 1 above the answer to the question is...

$$C_{2,3} = 10,000,000 \times \Bigg(1 + 0.25 \times \sin(1.2566 \times 1.25)\Bigg)^{-1} \times \Bigg(1.1298 + (0.25 \times -1.0569)\Bigg) = 6,925,000 \tag{18}$$

Appendix

A. We will define the following equations... [3]

$$E_1 = \text{Exp}\Bigg\{\alpha\, t\Bigg\} \text{ ...and... } E_2 = \text{Exp}\Bigg\{\alpha\, t\Bigg\} \sin(\beta\,(t + \phi)) \tag{19}$$

B. Using the first equation in Equation (19) above we will make the following integral definition... [3]

$$I(a,b)_1 = \int_a^b E_1\, \delta t = \text{Exp}\Bigg\{\alpha\, t\Bigg\}\, \alpha^{-1} \Bigg[_a^b \tag{20}$$

C. Using the second equation in Equation (19) above we will make the following integral definition... [3]

$$I(a,b)_2 = \int_a^b E_2\, \delta t = \text{Exp}\Bigg\{\alpha\, t\Bigg\}\Bigg(\alpha\, \sin(\beta\,(t + \phi)) - \beta\, \cos(\beta\,(t + \phi))\Bigg)\Bigg(\alpha^2 + \beta^2\Bigg)^{-1} \Bigg[_a^b \tag{21}$$

References

[1] Gary Schurman, *The Lognormal Distribution,* June, 2015.

[2] Gary Schurman, *An Introduction To Stochastic Calculus,* February, 2012.

[3] Gary Schurman, *Modeling The Business Cycle - Mathematical Supplement,* October, 2020.